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Feasible regions meets pattern avoidance
The long awaited 3rd of feasible regions
SFSU Algebra Geometry and Combinatorics Seminar

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Joint work with Jacopo Borga

Slides can be found in
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Patterns in permutations

A permutation π of size n is an arrangement on an $n \times n$ table:

$$\pi = \quad = 2431$$

Select a set I of columns of the square configuration of π and define the **restriction** $\pi|_I$. This is a permutation.

$$\pi|_{\{1,2,4\}} \qquad = \qquad = 231$$

Number of occurrences

We can count occurrences of each of the $k!$ permutations of size k in a big permutation σ.

For permutations π, σ, we define the pattern number:

$$\mathrm{occ}(\pi, \sigma) = \#\{\text{occurrences of } \pi \text{ in } \sigma\} .$$

In this way we have

$$\mathrm{occ}(12, 4132) = 2, \ \mathrm{occ}(312, 4132) = 2, \ \mathrm{occ}(12, 12345) = 10$$

$$\text{and} \ \ \mathrm{occ}(312, 3675421) = 0$$

$$\widetilde{\mathrm{occ}}(\pi, \sigma) = \frac{\mathrm{occ}(\pi, \sigma)}{\binom{|\sigma|}{|\pi|}}, \ \ \widetilde{\mathrm{occ}}_k(\sigma) = (\widetilde{\mathrm{occ}}(\pi, \sigma))_{\pi \in \mathcal{S}_k} \in \mathbb{R}^{\mathcal{S}_k} .$$

Plotting these relationships

For a fixed integer k, what are the possible values of $(\widetilde{\mathrm{occ}}(\pi, \sigma))_{\pi \in \mathcal{S}_k}$ when $|\sigma|$ is big?
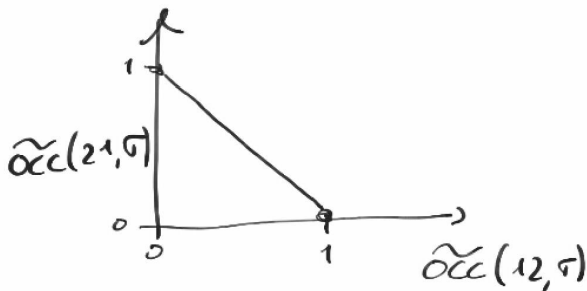

Figure: The interplay between proportion of occurrences of 12 and 21.

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Feasible region - Classical patterns

For a fixed integer k, the corresponding feasible region (FReg) is defined as follows

$$F_k := \{\vec{v} \in \mathbb{R}^{\mathcal{S}_k} \,|\, \exists\, \sigma^{(n)}, \widetilde{\mathrm{occ}}_k(\sigma^{(n)}) \to \vec{v}, |\sigma^{(n)}| \to \infty\}\,.$$

$F_{\leq k}$ - the FReg indexed by all permutations of size at most k
$F_{\mathcal{S}}$ - the FReg indexed by a set of permutations $\mathcal{S}$.

$F_{\{\pi\}}$ - an interval and is often studied in the context of *packing problems*.
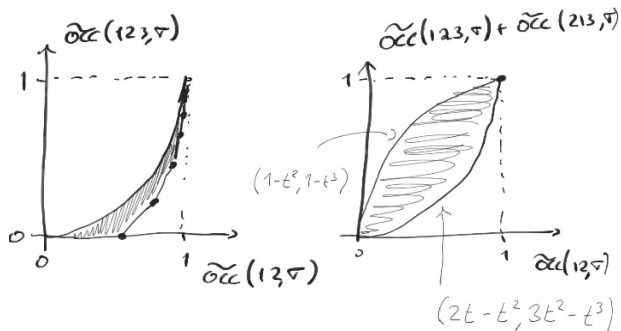
Feasible region - Examples

Figure: **Left:** The FReg comparing 12 and 123. **Right:** The FReg comparing patterns of 12 and patterns of 123 or 213 becomes a scalloped triangle. Feasible regions due to Kenyon, Kral, et.al. 2015.

Feasible region - The dimension problem

Theorem (Glebov, Hoppen, et.al. 2017)

The dimension of the feasible region $F_{\leq k}$ is at least the number of indecomposable permutations of size k.

Theorem (Vargas, 2014)

*The feasible region $F_{\leq k}$ satisfies a set of algebraic equations indexed by the **Lyndon permutations** of size up to k.*

Conjecture

*The codimension of the feasible region $F_{\leq k}$ is precisely the number of **Lyndon permutations** of size up to k.*

Consecutive occurrences

We now consider only occurrences that form **an interval**. For instance, taking $\sigma = 2413$, there are two distinct consecutive restrictions of σ of size three, namely 231 and 312.

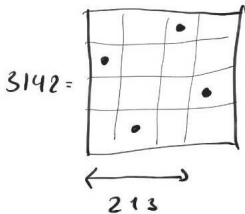$$\text{c-occ}(\pi, \tau) = \#\{I \text{ interval s.t. } \tau|_I = \pi\}.$$

Figure: The permutation 3142, does not contain a consecutive occurrence of 231, but it does contain a consecutive occurrence of 213.
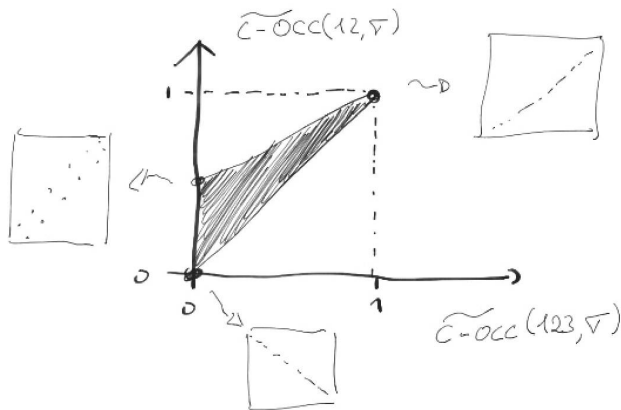
Consecutive occurrences

The number $\text{c-occ}(\pi, \sigma)$ varies between 0 and $|\sigma| - |\pi| + 1$. So we define

$$\widetilde{\text{c-occ}}(\pi, \sigma) = \frac{\text{c-occ}(\pi, \sigma)}{|\sigma|}, \;\; \widetilde{\text{c-occ}}_k(\sigma) = (\widetilde{\text{c-occ}}(\pi, \sigma))_{\pi \in \mathcal{S}_k} \in \mathbb{R}^{\mathcal{S}_k} \;.$$

New feasible region

$$\mathcal{F}_k := \{\vec{v} \in \mathbb{R}^{\mathcal{S}_k} | \exists \sigma^{(n)}, \widetilde{\text{c-occ}}_k(\sigma^{(n)}) \to \vec{v}, |\sigma^n(n)| \to \infty\} \subseteq \mathbb{R}^{\mathcal{S}_k}.$$

This is a closed and convex region.

The overlap graph

Consider the case $k = 3$ and the permutation $\sigma = 2714365$.

$$2714365 \mapsto 231 - 312 - 132 - 213 - 132 \, .$$
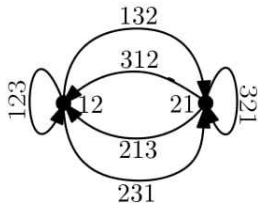
We can construct a graph from this:

Figure: The overlap graph for $k = 3$
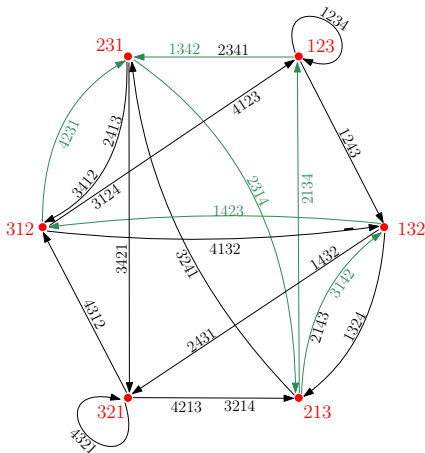
Another overlap graph

Figure: The overlap graph for $k = 4$, together with the path corresponding to $\sigma = 628451793$.
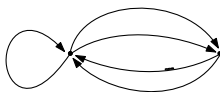
The cycle polytope

Figure: A graph with five cycles.

The cycle polytope is defined in $\mathbb{R}^{E(G)}$.

$$(\vec{e}_{\mathcal{C}})_x = \frac{1}{|\mathcal{C}|}\mathbb{1}[x \in \mathcal{C}]\,.$$

$$\mathrm{conv}\{\vec{e}_{\mathcal{C}}|\mathcal{C} \text{ is a simple cycle in } G\} \subseteq \mathbb{R}^{E(G)}\,.$$
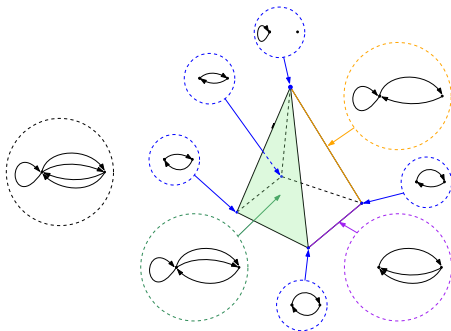
The cycle polytope

Figure: The overlap graph of a graph with five cycles.

The overlap graph - inverting a path

{ permutations } $\rightarrow$ { paths in $\mathcal{O}v(k)$ }, is this map invertible?

$$\omega = 2413 \rightarrow 4123 \rightarrow 1342 \rightarrow 2413.$$

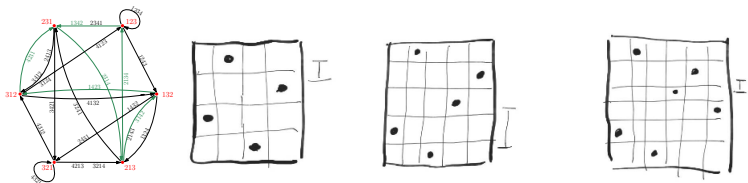Figure: The construction of the path ω.

It is a cycle polytope

Theorem (Borga, P., 2019)

$$P(\mathcal{O}v(k)) = \mathcal{F}_k .$$

In particular, $\mathcal{F}$ is a polytope with dimension $k! - (k-1)!$.
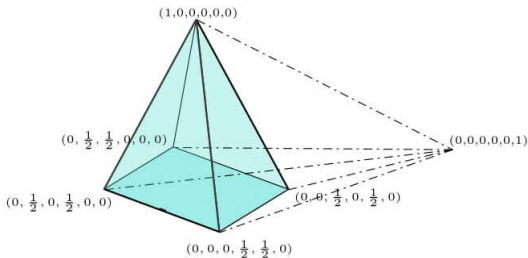

$(1,0,0,0,0,0)$

$(0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)$

$(0,0,0,0,0,1)$

$(0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0)$

$(0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0)$

$(0, 0, 0, \frac{1}{2}, \frac{1}{2}, 0)$

Figure: The feasible region of $k = 3$.

Avoiding set patterns - permutation classes

Let's introduce pattern avoidance in this problem!

$$\mathrm{Av}(\mathcal{P}) = \{\tau \in \mathcal{S} \,|\, \forall\, \pi \in \mathcal{P},\, \mathrm{occ}(\pi, \tau) = 0\}\,,$$
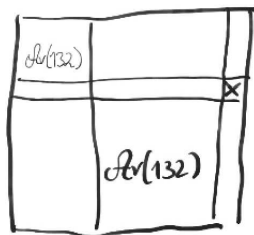
Let $\mathrm{Av}_k(\mathcal{P})$ be $\mathrm{Av}(\mathcal{P}) \cap \mathcal{S}_k$.

$$\mathrm{Av}(12) = \{1, 21\}, \# \,\mathrm{Av}_k(132) = C_k\,.$$
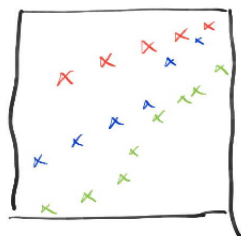
A set of the form $\mathrm{Av}(\mathcal{P}) \subseteq \mathcal{S}$ is called a **permutation class**.

Permutations classes are a world to be investigated!

Generating trees

Figure: **Left:** the permutation class $\mathrm{Av}(132)$ is characterized by in inductive construction. **Right:** the permutation classes $\mathrm{Av}(n \cdots 1)$ are characterized by $n - 1$ increasing monochromatic subsequences.

Does anyone read these titles?

The feasible region is:

$$\mathcal{F}_k^{\mathrm{Av}(\mathcal{P})} := \{\vec{v} \in \mathbb{R}^{\mathcal{S}_k} \,|\, \exists\, \sigma^{(n)} \in \mathrm{Av}(\mathcal{P}) \text{ with } \widetilde{\mathrm{c\text{-}occ}}_k(\sigma^{(n)}) \to \vec{v}\}.$$

$\mathcal{F}_k^{\mathrm{Av}(\mathcal{P})}$ is still a closed set. **Is it convex**?
Example: if $\mathcal{P} = \{132, 312, 231, 213\}$, then $\mathcal{F}_k^{\mathrm{Av}(\mathcal{P})}$ is a set with only two points.

Proposition
If $\mathcal{P}$ is a singleton, then $\mathcal{F}_k^{\mathrm{Av}(\mathcal{P})}$ is convex.

$$\{ \text{permutations in } \mathrm{Av}(\mathcal{P})\} \to \{ \text{paths in } \mathcal{O}v(k) \text{ avoiding } \mathcal{P} \}$$

Thus, $\mathcal{F}_k^{\mathrm{Av}(\mathcal{P})} \subseteq P(\mathcal{O}v(k)|_{\mathrm{Av}(\mathcal{P})})$.

Example of path inversion - 132

On the case $\mathcal{P} = \{132\}$, can we always invert such paths?
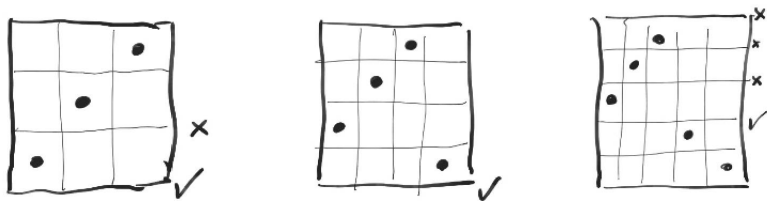Example:

$$\omega = 123 \to 231 \to 321 \to 213\,.$$

Figure: The construction of a permutation corresponding to the path ω.

The upshot - 132

$$\mathcal{F}_k^{\mathrm{Av}(132)} = P(\mathcal{O}v(k)|_{\mathrm{Av}_k(132)}) \text{ and } \dim \mathcal{F}_k^{\mathrm{Av}(132)} = C_k - C_{k-1}\,.$$
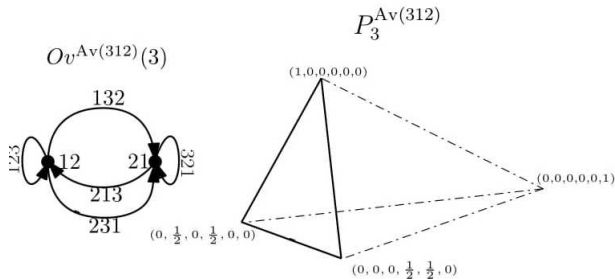

Figure: **Left:** The restricted overlap graph for $\mathcal{P} = \{312\}$. **Right:** The restricted feasible region for $k = 3$ and $\mathcal{P} = \{312\}$.

The overlap graph - 321

On the case 321, can we always invert such paths? Example:

$$\omega = 312 \rightarrow 123 \rightarrow 231 \,.$$

Recall: 321 avoiders have a monotone bicoloring. Let's add colours to the path, in such a way that each color is a monotone sequence:

$$\omega = 312 \rightarrow 123 \rightarrow 231 \,.$$

Incolorable!

The coloured overlap graph - 321

On the other hand, a valid path of colored permutations would be, for instance

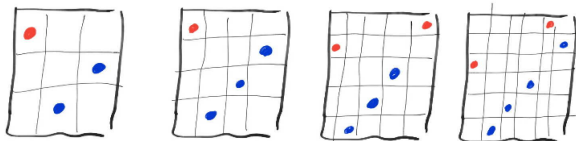$$\omega = 312 \rightarrow 123 \rightarrow 123 \rightarrow 132\,.$$

Figure: The construction of a permutation corresponding to the corrected path ω.

The coloured overlap graph - 321

Let's add colours to the overlap graph itself and call it
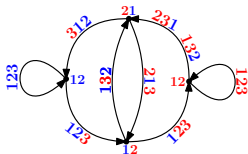$\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(321)}(k)$

Figure: The overlap graph for $k = 3$ adapted to $\mathcal{P} = \{321\}$, where now
we include all possible colouring of each edge.

Theorem (Borga, P. 2020)

$$\mathcal{F}_k^{\mathrm{Av}(n\cdots1)} = \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(n\cdots1)}(k))),$$

$$\dim \mathcal{F}_k^{\mathrm{Av}(n\cdots1)} = |\operatorname{Av}_k(n\cdots1)| - |\operatorname{Av}_{k-1}(n\cdots1)|.$$
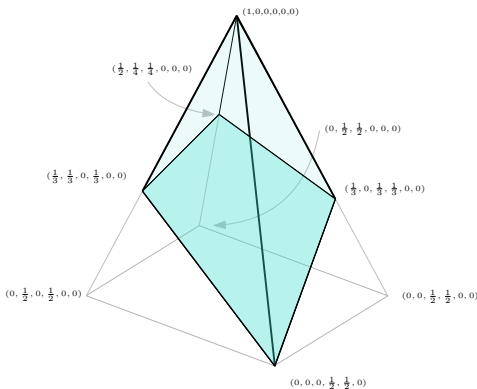
The restricted feasible region - 321

Figure: **Left:** $P(\mathcal{O}v(3))$. **Right:** The restricted feasible region for $k = 3$ and $\mathcal{P} = \{321\}$, overlaid with $P(\mathcal{O}v(3)|_{\mathrm{Av}_3(321)})$.

Related problems

- Other permutation classes are also given by generating trees. We believe that any such permutation class will have a direct description of the feasible region, and that we can totally describe all the extremal points.

- The dimension conjecture: if $\mathcal{P}$ has only one pattern, then

$$\dim \mathcal{F}_k^{\mathrm{Av}_k(\mathcal{P})} = |\mathrm{Av}_k(\mathcal{P})| - |\mathrm{Av}_{k-1}(\mathcal{P})|.$$

- The other dimension conjecture on the classical FReg.
- What is the volume of all these regions?

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